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Mixed variational formulations for continua with microstructure

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Abstract

In this paper we obtain mixed variational formulations for the fully linear elastic equilibrium problem of continua with vectorial microstructure by the application of the semi-inverse method proposed by Ji-Huan He. Applications to microcracked and piezoelectric bodies are then investigated.

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1. Introduction

We use the He's (1997a,b,c) *semi-inverse method* to obtain a variational characterization of weak solutions for the equilibrium problem of a linear hyperelastic body endowed with a vectorial microstructure.

For a *continuum with microstructure* we intend, in the manner of Capriz (1985, 1989, 2000), a "microstructured" body which can be modelled by using two (or perhaps more) descriptors of the physical configurations, namely the placement field (assigning to each material element its place in the Euclidean point space) and an *order parameter* field by which we want to account for the microstructure which characterizes the material comprising the body. Thus, behind the kinematical descriptor of classical continuum mechanics, the placement field or, equivalently, the displacement field, in the so called *multifield theories* there are new descriptors, the order parameters, which accounts for the presence of the microstructure. In the case of a vectorial microstructure the order parameter is a vector, called the *director*. All the descriptor fields are observable "objects" and then new measures of interaction behind the standard stress need to be associated to them and appropriately balanced.

The semi-inverse method was first introduced by He (1997c) in variational formulations of fluid mechanics. [1] By the application of the semi-inverse method one could obtain a variational characterization of

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[1] He's semi-inverse method was first proposed in his Ph.D. Thesis "A New Approach to Establishing Generalized Variational Principles and C.C. Lin's Constraints", Shanghai University, 1997.

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a differential problem starting directly from the system of differential equations and boundary conditions which describe the problem itself. This fact allows to have several different variational formulations of the same problem, very useful in technical and computational applications. As pointed out by He, it therefore allows to overcome the so called *variational crisis*. This phenomenon was studied by different authors: it is well known that the traditional way to generalize variational principles is the Lagrange multipliers method; sometimes, however, during the process of multiplier identification some of them become zero, a circumstance which happens when a variable involved only in the constraint equations doesn't appear in the original functional one wants to generalize.

The semi-inverse method was first suggested by He (1997a,b) in fluid mechanics to establish generalized variational principles and in linear elasticity where it was proposed to obtain an equivalent formulation of the Hellinger–Reissner and Hu–Washizu principles; then the method was applied to rederive the variational principle of Hellinger–Reissner via a procedure different from the Lagrange multipliers. A generalization of the Hellinger–Reissner principle by the construction of a family of mixed functionals was obtained (He, 2000). This version of the theorem is very useful in the applications with the mixed finite element method. Finally, the method was used to obtain variational principles for electroelastic bodies (see He, 2001).

As we have already observed the procedure used by He could be generalized using more general trial functionals, characterized by a Lagrange function totally unknown which satisfies only the integrability condition (see Mosconi, 2002).

In this paper we want to obtain by means of the semi-inverse method a mixed variational formulation which is a generalization of the Hu–Washizu principle, in the case of a linear elastic "microstructured" continuum endowed with an order parameter field of vector type. The six fields mixed functional we get is more general than the linear ones proposed in (Steinmann and Stein, 1997) for micropolar continua. The proof is given in the case of a vectorial order parameter but the same procedure applies also for a generic order parameter.

Based on the obtained variational principles, various numerical methods (finite element methods, meshfree methods) can be powerfully applied and various approximate theories for linear elastic plates and rods with microstructure, useful in engineering applications, can be derived.

2. Mathematical formulation of equilibrium for continua with microstructure

Here we recall the basic equations of the theory of continua with vectorial microstructure [2] (for general treatments see Capriz (1989, 2000) and Mariano (2001)).

Let $B = B(t)$, a regular bonded open subregion of three-dimensional Euclidean point space $\mathscr{E}$, be the *current* configuration at a given time t of a continuous body $\mathscr{B}$ endowed with a microstructure and $B_\mathrm{R} = B(t_0)$ a fixed *referential* configuration of the body. The *material element* p of such a body $\mathscr{B}$, whose reference placement we denote with $\mathbf{X}$, can be seen as a Lagrangean system, identified by the current placement $\mathbf{x} \in \mathscr{E}$ of its centre of mass (the *apparent placement*) plus extra degrees of freedom, a finite number (for example m) of independent Lagrangean variables (the order parameters) which describe the microstructure. It is possible to choose the order parameters in an element d of an appropriate differentiable manifold $\mathscr{M}$ of dimension m. So the *complete placement* of a material element p is the following mapping of $\mathbb{R} \times B_\mathrm{R} \subset \mathscr{E}$ into $\mathscr{E} \times \mathscr{M}$:

$$(\mathbf{x}(\mathbf{X}, t), \mathsf{d}(\mathbf{X}, t)), \tag{2.1}$$

[2] We will denote by $\mathscr{V}$ the vector space associated to the point space $\mathscr{E}$. We use SanSerif letters to denote microstructural quantities, so the microstructural fields T, b shall not be confused with the macroscopic ones $\mathbf{T}$, $\mathbf{b}$.

in the sequel we will neglect the time variable and we shall assume that $\mathbf{x}(\cdot)$ and $\mathsf{d}(\cdot)$ are continuous and piecewise continuously differentiable in space. Moreover, we will consider only the case of a vectorial order parameter d, from now on the director, and we will write $\nabla\mathsf{d}$ for the gradient of the director, [3] $\mathbf{u} = \mathbf{x}(\mathbf{X}) - \mathbf{X}$ for the displacement field and $\mathbf{E}$ for the infinitesimal strain tensor associated to the macro-displacement. In this work we will consider only infinitesimal mechanical deformations ($\|\nabla\mathbf{u}\| = \alpha_1$, with α_1 a small parameter), so reference and current configurations will be confused for each other.

In the model of continua with vectorial microstructure (a special case of the general format given by Capriz (1989)) the following fields must be considered in order to formulate the equilibrium problem of such a body:

- the current stress $\mathbf{T}$,
- the volume force $\mathbf{b}$,
- the boundary force $\mathbf{s}$,
- the microstress T,
- the volume microforce b,
- the boundary microforce s,
- the interactive microforce k,

The last vector field k (also called internal self-force) is a term which accounts for the internal interactions in the substructure, while the vector field b accounts for the external volume forces acting upon the substructure; in the general theory with a generic order parameter, they are both elements of the *cotangent space* $\mathcal{T}^*_{\mathsf{d}}\mathcal{M}$ of $\mathcal{M}$ at d. The microstress T is a linear transformation of $\mathcal{V}$ to $\mathcal{T}^*_{\mathsf{d}}\mathcal{M}$, in this case an element of Lin, [4] such that a generalization of the classical Cauchy theorem gives: $\mathsf{s} = \mathsf{T}\mathbf{n}$ (see Capriz and Virga, 1990), which is in general the element of the cotangent space, in the case a vector too, representing the action exerted on the microstructure through a surface element whose unit exterior normal is $\mathbf{n}$ (it is called the generalized traction, or briefly, the microtraction and physically it accounts for the interaction between neighboring substructures).

We suppose that the boundary ∂B of the body is composed of two sets of two complementary and disjoint portions, namely:

$$\partial_1 B \cup \partial_2 B = \partial B, \quad \partial_1 B \cap \partial_2 B = \emptyset,$$
$$\partial_3 B \cup \partial_4 B = \partial B, \quad \partial_3 B \cap \partial_4 B = \emptyset,$$

and that in each point of ∂B the outward unit normal $\mathbf{n}$ is well-defined. Moreover, the following data are given: a system of loads, i.e. two pairs $(\mathbf{b}, \mathbf{s}_0)$ and $(\mathsf{b}, \mathsf{s}_0)$, where $\mathbf{b}$ is the volume–load vector field and b the micro-volume–load vector field both defined over B, $\mathbf{s}_0$ and s_0 being respectively the surface–load vector defined over $\partial_2 B$ and the micro-surface–load vector defined over $\partial_4 B$ (they are data), and two fields $\mathbf{u}_0$ and d_0 assigned over $\partial_1 B$ and $\partial_3 B$ respectively.

If constraints are applied to the substructure in the same portion $\partial_1 B$ of the boundary in which the motion is constrained, then

$$\partial_1 B \equiv \partial_3 B \quad \text{and} \quad \partial_2 B \equiv \partial_4 B, \tag{2.2}$$

this particular case often occurs in physical applications.

[3] Because of we are concerning with the linear theory we do not make distinction between referential or current differential operators, even if we underline that the order parameter field (as all fields involved in mechanical models) may have a *spatial* $\mathsf{d}(\mathbf{x},t)$ and a *material* $\mathsf{d}(\mathbf{X},t)$ description.

[4] Lin denotes the space of second-order tensors, while sym the space of symmetric second-order tensors and skw the space of skew-symmetric second-order tensors, with Lin = Sym ⊕ Skw.

An elastic state for a continuum with vectorial microstructure (briefly, a vectorial "microstructured" elastic state) corresponding to the volume forces $(\mathbf{b}, \mathsf{b})$ is an admissible [5] ordered array of the form $(\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T})$ such that the following equations are fulfilled:

- equilibrium equations: [6]

$$\operatorname{div}\mathbf{T} + \mathbf{b} = 0 \quad \text{in } B, \tag{2.3a}$$
$$\operatorname{div}\mathsf{T} + \mathsf{b} + \mathsf{k} = 0 \quad \text{in } B, \tag{2.3b}$$
$$\operatorname{skw}(\mathbf{T} + \mathsf{T}\nabla\mathsf{d}^{\mathrm{T}} - \mathsf{k}\otimes\mathsf{d}) = 0 \quad \text{in } B, \tag{2.3c}$$
$$\mathbf{Tn} - \mathbf{s}_0 = 0 \quad \text{on } \partial_2 B, \tag{2.3d}$$
$$\mathsf{T}\mathbf{n} - \mathsf{s}_0 = 0 \quad \text{on } \partial_4 B, \tag{2.3e}$$

- kinematical equations:

$$\mathbf{E} - \operatorname{sym}\nabla\mathbf{u} = 0 \quad \text{in } B, \tag{2.4a}$$
$$\mathbf{u} - \mathbf{u}_0 = 0 \quad \text{on } \partial_1 B, \tag{2.4b}$$
$$\mathsf{d} - \mathsf{d}_0 = 0 \quad \text{on } \partial_3 B, \tag{2.4c}$$

- constitutive equations:

$$\begin{aligned} \operatorname{sym}\mathbf{T} &= \frac{\partial w}{\partial \mathbf{E}}, \\ \mathsf{k} &= -\frac{\partial w}{\partial \mathsf{d}}, \quad \text{in } B, \\ \mathsf{T} &= \frac{\partial w}{\partial(\nabla\mathsf{d})}. \end{aligned} \tag{2.5}$$

Eqs. (2.3a)–(2.3c) are, respectively, the standard Cauchy's balance of force, the balance of substructural interactions (or balance of microforce) and the torque balance.

Remark 1. We observe that, in the general theory, the balance of substructural interactions is not given only by Eq. (2.3b), because the element of the cotangent space $\operatorname{div}\mathsf{T} + \mathsf{b} + \mathsf{k}$ must belong to the null space of the linear operator A^{T}, where A is the infinitesimal generator of the local action on $\mathscr{M}$ of the group of rotations. When A is of the form $\mathsf{d}\times$ this null space coincides with the whole tangent space of $\mathscr{M}$ at d and the balance of microinteractions has the form (2.3b) (for the general theory justifying these assertions see Capriz and Virga (1990, 1994); Mariano (2001)).

[5] We recall that in the classical linear theory of elasticity an elastic state $(\mathbf{u}, \mathbf{E}, \mathbf{T})$ is admissible if (see Gurtin, 1972): (i) the vector field $\mathbf{u}$ is an admissible displacement field i.e. $\mathbf{u}$ is of class C^2 on B and $\mathbf{u}$ and $\operatorname{sym}\nabla\mathbf{u}$ are continuous on $\overline{B}$; (ii) $\mathbf{E}$ is an admissible strain field i.e. $\mathbf{E}\in\mathrm{Sym}$ and $\mathbf{E}$ is continuous on $\overline{B}$; (iii) $\mathbf{T}$ is an admissible stress field i.e. $\mathbf{T}\in\mathrm{Sym}$, $\mathbf{T}$ is smooth on B and $\mathbf{T}$ and $\operatorname{div}\mathbf{T}$ are continuous on $\overline{B}$. In the case of elasticity with microstructure it is well known that the stress tensor $\mathbf{T}$ could not be symmetric because of the presence of the microstructure itself, so for the definition of an admissible state for a continuum with microstructure $\Xi = (\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T})$, briefly an admissible "microstructured" state, one has to change the condition (iii) not requiring that $\mathbf{T}\in\mathrm{Sym}$ and has to add the following conditions for the fields relative to the microstructure; (iv) d is of class C^2 on B and continuous on $\overline{B}$; (v) $\nabla\mathsf{d}$ is continuous on $\overline{B}$; (vi) T is smooth on B and T and $\operatorname{div}\mathsf{T}$ are continuous on $\overline{B}$. The previous conditions (i)–(vi) are the simplest regularity conditions one has to require in order to give meaning to the equations we are going to consider.

[6] We will denote by $\operatorname{sym}\mathbf{A} = 1/2(\mathbf{A} + \mathbf{A}^{\mathrm{T}})$ and $\operatorname{skw}\mathbf{A} = 1/2(\mathbf{A} - \mathbf{A}^{\mathrm{T}})$ respectively the symmetric part and the skew-symmetric part of a second-order tensor $\mathbf{A}$. Moreover $\mathbf{a}\otimes\mathbf{b}$ will denote the tensor product, such that $(\mathbf{a}\otimes\mathbf{b})\mathbf{u} = (\mathbf{b}\cdot\mathbf{u})\mathbf{a}$, $\forall\mathbf{a}, \mathbf{b}, \mathbf{u}\in\mathscr{V}$.

The constitutive equations (2.5) are the general form of constitutive equations that one could obtain in the conservative case from the existence of an elastic potential, by the local dissipation inequality applied to an hyperelastic material with a response function of the form

$$w = \hat{w}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d}). \tag{2.6}$$

In the present paper we restrict our analysis to the case in which the order parameter is a vector (the director) belonging to $\mathbb{R}^3$. We assume also that the body undergoes small deformations and $\|\mathsf{d}\| = \alpha_2$ and $\|\nabla\mathsf{d}\| = \alpha_3$, with α_2, α_3 small parameters. By a *linearization* procedure with respect to $\alpha = \max\{\alpha_1, \alpha_2, \alpha_3\}$, assuming the existence of a generalized stress-free state considered as natural state of the body, we get from (2.5) the following constitutive equations

$$\operatorname{sym}\mathbf{T} = \mathbf{C}[\mathbf{E}] + \mathbf{h}[\mathsf{d}] + \mathbf{H}[\nabla\mathsf{d}], \tag{2.7a}$$

$$\mathsf{k} = -\mathbf{K}\mathsf{d} - \mathbf{h}^{\mathrm{T}}[\mathbf{E}] - \mathbf{f}[\nabla\mathsf{d}], \tag{2.7b}$$

$$\mathsf{T} = \mathbf{S}[\nabla\mathsf{d}] + \mathbf{H}^{\mathrm{T}}[\mathbf{E}] + \mathbf{f}^{\mathrm{T}}[\mathsf{d}], \tag{2.7c}$$

whereas the elastic energy (2.6) has the quadratic representation:

$$\hat{w}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d}) = \tfrac{1}{2}(\mathbf{C}[\mathbf{E}]\cdot\mathbf{E} + \mathbf{K}\mathsf{d}\cdot\mathsf{d} + \mathbf{S}[\nabla\mathsf{d}]\cdot\nabla\mathsf{d} + 2\mathbf{h}[\mathsf{d}]\cdot\mathbf{E} + 2\mathbf{H}[\nabla\mathsf{d}]\cdot\mathbf{E} + 2\mathbf{f}[\nabla\mathsf{d}]\cdot\mathsf{d}), \tag{2.8}$$

here $\mathbf{C}$, $\mathbf{S}$ and $\mathbf{H}$ are fourth-order tensors, the first two symmetric and positive-definite, $\mathbf{h}$ and $\mathbf{f}$ are third-order tensors and $\mathbf{K}$ is a second-order positive-definite symmetric tensor; in particular $\mathbf{h}$ maps $\mathscr{V}$ into Sym while $\mathbf{H}$ maps Lin into Sym. [7] The energy (2.8) coincides with the purely linear elastic energy density when the microstructure is absent, that is $\mathsf{d} = 0$.

A situation which is interesting for applications is when the energy (2.6) admits the decomposition

$$\hat{w}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d}) = \varphi(\mathbf{E}, \mathsf{d}) + \vartheta(\nabla\mathsf{d}), \tag{2.9}$$

in such a way that

$$\operatorname{sym}\mathbf{T} = \frac{\partial\varphi}{\partial\mathbf{E}}, \quad \mathsf{k} = -\frac{\partial\varphi}{\partial\mathsf{d}}, \quad \mathsf{T} = \frac{\partial\vartheta}{\partial(\nabla\mathsf{d})}, \tag{2.10}$$

where

$$\varphi(\mathbf{E}, \mathsf{d}) = \tfrac{1}{2}(\mathbf{C}[\mathbf{E}]\cdot\mathbf{E} + \mathbf{K}\mathsf{d}\cdot\mathsf{d} + 2\mathbf{h}[\mathsf{d}]\cdot\mathbf{E}). \tag{2.11}$$

The term

$$\vartheta(\nabla\mathsf{d}) = \tfrac{1}{2}\mathbf{S}[\nabla\mathsf{d}]\cdot\nabla\mathsf{d} \tag{2.12}$$

is the *interfacial energy* and it represents the energy contribution of the weakly non-local interactions due to neighboring substructures. The assumption of the decomposition (2.9) is equivalent to the physical consideration that the variation of the global energy due to the variation of the microstructure is a somewhat local term restricted to the interface between neighboring substructures. The constitutive equations obtained are very common for the applications in electromagnetic solids mechanics. In Section 5 we will discuss another simplified version of constitutive equations (2.7a)–(2.7d) used to model microcracked bodies in (Mariano and Stazi, 2001), whereas in Section 6 we shall describe piezoelectric bodies as an example of continua with a scalar microstructure.

[7] Accordingly, $\mathbf{C}$ has the two minor symmetries while $\mathbf{H}$ and $\mathbf{h}$ the minor symmetry relative to the first two indices (left minor symmetry).

Remark 2. We observe that the torque balance (2.3c) represents a constitutive prescription on the stress field $\mathbf{T}$; in fact, it follows from the axiom of invariance of the internal power under changes of observer. In the linear case considered here, as a consequence of the linearization, the balance (2.3c) reduces to the following symmetry condition on the stress $\mathbf{T}$:

$$\operatorname{skw}\mathbf{T} + o(\alpha) = 0. \tag{2.13}$$

Thus from the constitutive equation (2.7a) we get the linear elastic response function in term of the total Cauchy stress, namely

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d}) = \mathbf{C}[\mathbf{E}] + \mathbf{h}[\mathsf{d}] + \mathbf{H}[\nabla\mathsf{d}], \tag{2.7d}$$

and in the definition of an admissible linear elastic state for the considered "microstructured" continuum (cf. Footnote 5, condition (iii)), one should require that $\mathbf{T} \in \mathrm{Sym}$. In the sequel we will denote by $\mathscr{A}$ the space of all admissible vectorial "microstructured" linear elastic states.

We have just summarized briefly some basic results from the theory of materials with elastic substructures, restricting our analysis to some cases of linear elastic behavior of models with vectorial order parameters. For the sake of brevity, we will denote by $\mathscr{S}$ the system of equations (2.3a), (2.3b), (2.3d), (2.3e) $\cup$ (2.4a)–(2.4c) $\cup$ (2.7d), (2.7b), (2.7c) $\cup$ (2.13).

Let us define the following *mixed* real functional:

$$\Gamma : \mathscr{A} \to \mathbb{R},$$

$$\begin{aligned}\Gamma\{\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T}\} = &\int_B [\hat{w}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d}) + \mathbf{T}\cdot(\operatorname{sym}\nabla\mathbf{u} - \mathbf{E}) - \mathbf{b}\cdot\mathbf{u} - \mathsf{b}\cdot\mathsf{d}] - \int_{\partial_1 B} \mathbf{T}\mathbf{n}\cdot(\mathbf{u} - \mathbf{u}_0) \\ &- \int_{\partial_2 B} \mathbf{s}_0\cdot\mathbf{u} - \int_{\partial_3 B} \mathsf{T}\mathbf{n}\cdot(\mathsf{d} - \mathsf{d}_0) - \int_{\partial_4 B} \mathsf{s}_0\cdot\mathsf{d}\end{aligned} \tag{2.14}$$

with the energy density $\hat{w}(\mathbf{E}, \mathsf{d}, \nabla\mathsf{d})$ given by (2.8).

In Section 3 we will derive this functional from the system $\mathscr{S}$ by means of the semi-inverse method and we will prove the following variational characterization of weak solutions for the equilibrium problem of a continuum with a material microstructure described by a vector order parameter:

Proposition. *An ordered array* $\Xi = (\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla d, \mathsf{T})$ *is a linear elastic state for a continuum with vectorial microstructure if and only if it renders stationary the functional* (2.14), *i.e.*

$$\begin{aligned}&\delta\Gamma\{\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T}\}[\mathbf{v}, \mathbf{V}, \mathbf{R}, \mathsf{v}, \nabla\mathsf{v}, \mathsf{R}] = 0,\\ &\forall(\mathbf{v}, \mathbf{V}, \mathbf{R}, \mathsf{v}, \nabla\mathsf{v}, \mathsf{R}) \textit{ in the space of all admissible variations such that}\\ &\mathbf{V} = \mathit{sym}\,\nabla\mathbf{v}.\end{aligned} \tag{2.15}$$

We will discuss in the following how to construct the space of admissible variations.

3. Variational formulation by the semi-inverse method

We seek for a mixed variational characterization of the system $\mathscr{S}$ using the semi-inverse method, as suggested by He (2001) in the case of piezoelectricity.

Let us consider the following real functional

$$\Gamma\{\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T}\} = \int_B f\,\mathrm{d}v + Y \tag{3.1}$$

where f, the trial Lagrange function, is a smooth enough unknown function of the independent variables $\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T}$ and Y is the boundary integral. A functional like (3.1) is called a *trial functional*. One could construct various energy-like trial functionals to start from; (3.1) is the most general trial functional because there are no a priori hypotheses on the form of the function f. The function f must be determined by the equivalence between the stationarity conditions of the trial functional (named trial Euler equations) and the field equations of the system $\mathscr{S}$.

The stationarity condition of (3.1) with respect to the variable $\mathbf{u}$ gives

$$\delta_{\mathbf{u}}\Gamma = \int_B (\partial_{\mathbf{u}} f - \operatorname{div}(\partial_{\nabla\mathbf{u}} f)) \cdot \delta\mathbf{u} + \int_{\partial B} (\partial_{\nabla\mathbf{u}} f)\mathbf{n} \cdot \delta\mathbf{u} = 0, \quad \forall\, \delta\mathbf{u}, \tag{3.2}$$

where

$$\delta_{\mathbf{u}} f = \partial_{\mathbf{u}} f - \operatorname{div}(\partial_{\nabla\mathbf{u}} f) \tag{3.3}$$

is the *functional derivative* of f with respect to $\mathbf{u}$, which reduces to the partial derivative when f is not an explicit function of $\nabla\mathbf{u}$.

From (3.2) by localization on B we obtain the following trial Euler equation:

$$\partial_{\mathbf{u}} f - \operatorname{div}(\partial_{\nabla\mathbf{u}} f) = 0 \quad \text{in } B; \tag{3.4}$$

by the equivalence condition between (3.4) and the balance of force (2.3a) one obtains the following partial identification of f

$$f = \mathbf{T} \cdot \nabla\mathbf{u} - \mathbf{b} \cdot \mathbf{u} + f_1. \tag{3.5}$$

Substituting (3.5) into (3.1) and taking the variation with respect to $\mathbf{T}$ together with (2.13) yield the following trial Euler equation

$$\operatorname{sym}\nabla\mathbf{u} + \partial_{\mathbf{T}} f_1 = 0 \quad \text{in } B \tag{3.6}$$

and the equivalence condition with the kinematical equation (2.4a) gives

$$f_1 = -\mathbf{T} \cdot \mathbf{E} + f_2, \tag{3.7}$$

so (3.5) becomes

$$f = \mathbf{T} \cdot (\operatorname{sym}\nabla\mathbf{u} - \mathbf{E}) - \mathbf{b} \cdot \mathbf{u} + f_2. \tag{3.8}$$

Operating in the same manner, the stationary condition of (3.1) with f given by (3.8) with respect to $\mathbf{E}$ yields

$$-\mathbf{T} + \partial_{\mathbf{E}} f_2 = 0 \quad \text{in } B. \tag{3.9}$$

Integrating the equivalence condition between (3.9) and the constitutive equation (2.7d) we have

$$f_2 = \tfrac{1}{2}\mathsf{C}[\mathbf{E}] \cdot \mathbf{E} + \mathsf{h}[\mathsf{d}] \cdot \mathbf{E} + \mathsf{H}[\nabla\mathsf{d}] \cdot \mathbf{E} + f_3.$$

Now let us consider the variation of the functional Γ with respect to the director d:

$$\delta_{\mathsf{d}}\Gamma = \int_B (\partial_{\mathsf{d}} f - \operatorname{div}(\partial_{\nabla\mathsf{d}} f)) \cdot \delta\mathsf{d} + \int_{\partial B} (\partial_{\nabla\mathsf{d}} f)\mathbf{n} \cdot \delta\mathsf{d} = 0, \quad \forall\, \delta\mathsf{d}; \tag{3.10}$$

in the above condition there is the functional derivative of f with respect to the director and a term which will influence the boundary part of the functional, just as it happens for the variation $\delta\mathbf{u}$. The trial Euler equation one obtains by localization of (3.10) in B is

$$\partial_{\mathsf{d}} f - \operatorname{div}(\partial_{\nabla\mathsf{d}} f) = 0, \tag{3.11}$$

the substitution of the expression of f one has found yields

$$\partial_{\mathsf{d}} f_3 + \mathbf{h}^{\mathrm{T}}[\mathbf{E}] - \mathrm{div}(\partial_{\nabla \mathsf{d}} f_3) = 0. \tag{3.12}$$

The equivalence condition of (3.12) to the equation obtained by subtracting (2.3b) to (2.7b) yields

$$f_3 = \mathsf{T} \cdot \nabla \mathsf{d} + \tfrac{1}{2}\mathbf{K}\mathsf{d} \cdot \mathsf{d} + \mathbf{f}[\nabla \mathsf{d}] \cdot \mathsf{d} - \mathsf{b} \cdot \mathsf{d} + f_4. \tag{3.13}$$

The trial Euler equation for $\delta \nabla \mathsf{d}$ reads

$$\mathbf{H}^{\mathrm{T}}[\mathbf{E}] + \mathbf{f}^{\mathrm{T}}[\mathsf{d}] + \mathsf{T} + \partial_{\nabla \mathsf{d}} f_4 = 0 \tag{3.14}$$

and its equivalence to the constitutive equation (2.7c)

$$\mathbf{S}[\nabla \mathsf{d}] + \mathbf{H}^{\mathrm{T}}[\mathbf{E}] + \mathbf{f}^{\mathrm{T}}[\mathsf{d}] - \mathsf{T} = 0$$

gives

$$f_4 = \tfrac{1}{2}\mathbf{S}[\nabla \mathsf{d}] \cdot \nabla \mathsf{d} - 2\mathsf{T} \cdot \nabla \mathsf{d} + f_5. \tag{3.15}$$

By the same manipulation one has for $\delta \mathsf{T}$:

$$-\nabla \mathsf{d} + \partial_{\mathsf{T}} f_5 = 0 \Rightarrow f_5 = \mathsf{T} \cdot \nabla \mathsf{d}; \tag{3.16}$$

therefore, by (2.8) we obtain the following final expression of the Lagrange function:

$$f = \hat{w}(\mathbf{E}, \mathsf{d}, \nabla \mathsf{d}) + \mathbf{T} \cdot (\mathrm{sym}\, \nabla \mathbf{u} - \mathbf{E}) - \mathbf{b} \cdot \mathbf{u} - \mathsf{b} \cdot \mathsf{d}. \tag{3.17}$$

If we substitute (3.17) into the functional (3.1), the functional we obtain is still subjected to the boundary constraints (2.3d), (2.3e), (2.4b) and (2.4c). To incorporate these conditions into the functional and identify the boundary part Y of Γ we repeat the procedure by using the above method. To begin with we may express the boundary integral Y in the form

$$Y = \sum_{i=1}^{4} \int_{\partial_i \mathrm{B}} g_i \, \mathrm{d}a, \tag{3.18}$$

where g_i $(i = 1, \ldots, 4)$ are unknown functions, smooth enough, to be determined.

At the boundary of the gross structure it is possible to assign the values of the displacement (**u**) and of the traction ($\mathbf{s} = \mathbf{Tn}$); by analogy, for the substructure we suppose that one could assign at the boundary the values of the director (d) and of the work conjugate quantity, the generalized traction ($\mathsf{s} = \mathsf{T}\mathbf{n}$). So, without loss of generality, we may assume the following hypothesis

$$Y = Y\{\mathbf{u}, \mathbf{T}, \mathsf{d}, \mathsf{T}\}, \tag{3.19}$$

which obviously reflects into

$$g_i = g_i(\mathbf{u}, \mathbf{T}), \quad i = 1, 2, \tag{3.20}$$

$$g_i = g_i(\mathsf{d}, \mathsf{T}), \quad i = 3, 4. \tag{3.21}$$

By considering during the previous procedure the boundary integral as a functional $Y = Y\{\mathbf{u}, \mathbf{T}, \mathsf{d}, \mathsf{T}\}$ and taking into account also its variations produced by $\delta\mathbf{u}$, $\delta\mathbf{T}$, $\delta\mathsf{d}$, $\delta\mathsf{T}$, we obtain, together with the trial Euler equations considered before, the following trial stationary conditions on $\partial_i B$:

$$\delta \mathbf{u} \Rightarrow \mathbf{Tn} + \partial_{\mathbf{u}} g_i = 0, \quad i = 1, 2, \tag{3.22}$$

$$\delta \mathbf{T} \Rightarrow \partial_{\mathbf{T}} g_i = 0, \quad i = 1, 2, \tag{3.23}$$

$$\delta \mathsf{d} \Rightarrow \mathsf{T}\mathbf{n} + \partial_{\mathsf{d}} g_i = 0, \quad i = 3, 4, \tag{3.24}$$

$$\delta \mathsf{T} \Rightarrow \partial_{\mathsf{T}} g_i = 0, \quad i = 3, 4. \tag{3.25}$$

We consider first $\partial_1 B$, and thus conditions (3.22) and (3.23) written for $i = 1$ should be equivalent to the boundary kinematical condition (2.4b), which gives

$$g_1 = -\mathbf{T}\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_0). \tag{3.26}$$

Considering now $\partial_2 B$, conditions (3.22) and (3.23) written for $i = 2$ should be equivalent to the boundary static condition (2.3d) and this yields to

$$g_2 = -\mathbf{s}_0 \cdot \mathbf{u}. \tag{3.27}$$

In the same way, the equivalence condition between (3.24) and (3.25) written for $i = 3$ and the boundary condition (2.4c) yields to the following identification on $\partial_3 B$

$$g_3 = -\mathsf{T}\mathbf{n} \cdot (\mathsf{d} - \mathsf{d}_0) \tag{3.28}$$

and finally the same manipulation between (3.24) and (3.25) written for $i = 4$ and the boundary condition (2.3e) gives on $\partial_4 B$

$$g_4 = -\mathsf{s}_0 \cdot \mathsf{d}. \tag{3.29}$$

Substituting (3.17), (3.26)–(3.29) into the functional (3.1) with (3.18) we obtain the functional (2.14). Of course, if the influence of the material microstructure is negligible and one disregards the terms relative to the microstructure, then the obtained functional reduces to the well known Hu–Washizu functional (see Washizu, 1975).

4. The space of variations

In this section we calculate the Gâteaux derivative of the functional (2.14) to prove the proposition stated in Section 2. This simple calculation leads to the identification of the space of admissible variations for the states space $\mathscr{A}$ one has to consider when deals with the proposed generalized Hu–Washizu-like functional.

Let us consider a variation of the stress field $\mathbf{T} + \varepsilon\mathbf{R}$ and of the microstress field $\mathsf{T} + \varepsilon\mathsf{R}$, with $\varepsilon \in \mathbb{R}$; these fields should satisfy the equilibrium equations (2.3a), (2.3b) and (2.13); thus we have the following characterization of the space $\mathscr{W}$ of all admissible variations: $\mathscr{W}$ is the set of all ordered array of regular fields $(\mathbf{v}, \mathbf{V}, \mathbf{R}, \mathsf{v}, \nabla\mathsf{v}, \mathsf{R}) \in \mathscr{A}$ such that

$$\begin{aligned} &\mathbf{V} = \operatorname{sym}\nabla\mathbf{v}, \\ &\mathbf{v} = 0, \ \text{on } \partial_1 B, \quad \mathsf{v} = 0, \text{on } \partial_3 B, \\ &\operatorname{div}\mathbf{R} = 0, \quad \operatorname{div}\mathsf{R} = 0, \text{in } B, \\ &\mathbf{R}\mathbf{n} = 0, \ \text{on } \partial_2 B, \quad \mathsf{R}\mathbf{n} = 0, \text{on } \partial_4 B. \end{aligned} \tag{4.1}$$

If we now differentiate at $\varepsilon = 0$ the mapping

$$\varepsilon \to \Gamma\{\mathbf{u} + \varepsilon\mathbf{v}, \nabla\mathbf{u} + \varepsilon\nabla\mathbf{v}, \mathbf{E} + \varepsilon\mathbf{V}, \mathbf{T} + \varepsilon\mathbf{R}, \mathsf{d} + \varepsilon\mathsf{v}, \nabla\mathsf{d} + \varepsilon\nabla\mathsf{v}, \mathsf{T} + \varepsilon\mathsf{R}\}, \tag{4.2}$$

by definition (2.8), conditions (4.1) and applying the divergence theorem, we arrive at the first variation of the functional (2.14) evaluated at $(\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{d}, \nabla\mathsf{d}, \mathsf{T})$:

$$\begin{aligned}\delta\Gamma\{\mathbf{u},\mathbf{E},\mathbf{T},\mathsf{d},\nabla\mathsf{d},\mathsf{T}\}[\mathbf{v},\mathbf{V},\mathbf{R},\mathsf{v},\nabla\mathsf{v},\mathsf{R}] = \int_B &[(\operatorname{sym}\nabla\mathbf{u}-\mathbf{E})\cdot\mathbf{R}-(\operatorname{div}\mathbf{T}+\mathbf{b})\cdot\mathbf{v}\\ &+(\mathbf{C}[\mathbf{E}]+\mathbf{h}[\mathsf{d}]+\mathbf{H}[\nabla\mathsf{d}]-\mathbf{T})\cdot\mathbf{V}\\ &-(\operatorname{div}\mathsf{T}+\mathsf{b}-\mathbf{h}^{\mathrm{T}}[\mathbf{E}]-\mathbf{K}[\mathsf{d}]-\mathbf{f}[\nabla\mathsf{d}])\cdot\mathsf{v}\\ &+(\mathbf{S}[\nabla\mathsf{d}]+\mathbf{H}^{\mathrm{T}}[\mathbf{E}]+\mathbf{f}^{\mathrm{T}}[\mathsf{d}]-\mathsf{T})\cdot\nabla\mathsf{v}]\\ &-\int_{\partial_1 B}\mathbf{R}\mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_0)+\int_{\partial_2 B}(\mathbf{T}\mathbf{n}-\mathbf{s}_0)\cdot\mathbf{v}\\ &-\int_{\partial_3 B}\mathsf{R}\mathbf{n}\cdot(\mathsf{d}-\mathsf{d}_0)+\int_{\partial_4 B}(\mathsf{T}\mathbf{n}-\mathsf{s}_0)\cdot\mathsf{v}\end{aligned} \tag{4.3}$$

which proves the statement (2.15).

Remark 3. We observe that the localization on B of the stationary condition with respect to the variation v gives the following result

$$\operatorname{div}\mathsf{T}+\mathsf{b}=\mathbf{h}^{\mathrm{T}}[\mathbf{E}]+\mathbf{K}[\mathsf{d}]+\mathbf{f}[\nabla\mathsf{d}], \tag{4.4}$$

where the right term is an internal force which represents the interactions within the substructure; if we identify this term with the opposite of the interactive microforce k, from (4.4) we have at the same time both the balance of substructural interactions (2.3b) and the constitutive equation (2.7b).

The mixed functional (2.14) could be useful in performing numerical analysis by means of the finite element method. In this case another formulation consists into considering the following mixed functional defined over the subset $\mathscr{A}_c \subset \mathscr{A}$ of all kinematical admissible states, i.e. all admissible states which satisfy the compatibility condition (2.4a):

$$\Psi : \mathscr{A}_c \to \mathbb{R},$$

$$\begin{aligned}\Psi\{\mathbf{u},\mathbf{T},\mathsf{d},\nabla\mathsf{d},\mathsf{T}\} = \int_B[\breve{w}(\mathbf{u},\mathsf{d},\nabla\mathsf{d})-\mathbf{b}\cdot\mathbf{u}-\mathsf{b}\cdot\mathsf{d}]-\int_{\partial_1 B}\mathbf{T}\mathbf{n}\cdot(\mathbf{u}-\mathbf{u}_0)-\int_{\partial_2 B}\mathbf{s}_0\cdot\mathbf{u}\\ -\int_{\partial_3 B}\mathsf{T}\mathbf{n}\cdot(\mathsf{d}-\mathsf{d}_0)-\int_{\partial_4 B}\mathsf{s}_0\cdot\mathsf{d};\end{aligned} \tag{4.5}$$

the stationary condition of this Hellinger–Reissner-like functional represents the variational characterization of balance equations (2.3a), (2.3b), (2.3d), (2.3e), constitutive equations (2.7d), (2.7b), (2.7c) and kinematical boundary conditions (2.4b) and (2.4c).

If we also assume that constitutive equations (2.7d), (2.7b), (2.7c) and kinematical boundary conditions (2.4b) and (2.4c) are a priori fulfilled, we obtain from (4.5) the following total potential energy functional for a linear elastic body with a vectorial microstructure

$$\Phi\{\mathbf{u},\mathsf{d},\nabla\mathsf{d}\} = \int_B[\widehat{w}(\mathbf{u},\mathsf{d},\nabla\mathsf{d})-\mathbf{b}\cdot\mathbf{u}-\mathsf{b}\cdot\mathsf{d}]-\int_{\partial_2 B}\mathbf{s}_0\cdot\mathbf{u}-\int_{\partial_4 B}\mathsf{s}_0\cdot\mathsf{d}, \tag{4.6}$$

with

$$\widehat{w}(\mathbf{u},\mathbf{d},\nabla\mathsf{d}) = \tfrac{1}{2}(\widehat{\mathbf{T}}(\mathbf{u},\mathsf{d},\nabla\mathsf{d})\cdot\nabla\mathbf{u}+\widehat{\mathsf{T}}(\mathbf{u},\mathsf{d},\nabla\mathsf{d})\cdot\nabla\mathsf{d}-\widehat{\mathsf{k}}(\mathbf{u},\mathsf{d},\nabla\mathsf{d})\cdot\mathsf{d}), \tag{4.7}$$

which is a three fields functional whose stationary condition is the variational characterization of the force and microforce balance equations (2.3a) and (2.3b) and the associated static boundary conditions (2.3d) and (2.3e).

Remark 4. When the material microstructure which characterizes the behavior of a body can be described by a scalar-valued order parameter d, as in porous bodies or in two-phase systems (see Capriz, 1989; Mariano, 2001), the microstress is a vector-valued field T while the generalized traction $\mathbf{t}$ and the volume microforce b are scalar-valued. The previous format can be applied yet: the functional (2.14) accordingly transforms into the following:

$$\Theta\{\mathbf{u},\mathbf{E},\mathbf{T},d,\nabla d,\mathsf{t}\} = \int_B [\hat{w}(\mathbf{E},d,\nabla d) + \mathbf{T}\cdot(\operatorname{sym}\nabla\mathbf{u}-\mathbf{E}) - \mathbf{b}\cdot\mathbf{u} - \mathsf{b}d] - \int_{\partial_1 B}\mathbf{Tn}\cdot(\mathbf{u}-\mathbf{u}_0) - \int_{\partial_2 B}\mathbf{s}_0\cdot\mathbf{u} - \int_{\partial_3 B}(d-d_0)\mathsf{T}\cdot\mathbf{n} - \int_{\partial_4 B}\mathsf{s}_0 d \tag{4.8}$$

and (4.5) and (4.6) consequently transform.

5. Application I. Microcracked bodies

As pointed out by Mariano (1999) it is possible to study microcracked bodies in the context of multifield theories by using the model of continua with microstructure instead of the classical internal variable scheme.

In a body $\mathscr{B}$ with diffused microcracks inside, under any deformation mapping a material point p into a point $\mathbf{x}$ of $\mathscr{E}$, the patch at the reference position $\mathbf{X}$ undergoes a (complete) displacement $\mathbf{u}(\mathbf{X}) + \mathsf{u}(\mathbf{X})$

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}) + \mathsf{u}(\mathbf{X}), \tag{5.1}$$

where the order parameter $\mathsf{u}(\mathbf{X})$ represents the kinematical component to the displacement field due to the enlargement or the closure of microcracks enclosed in a neighborhood of $\mathbf{X}$; $\mathsf{u}(\mathbf{X})$ is called the mean relative displacement and it represents the difference between the effective (real) displacement of the patch at $\mathbf{X}$ and the theoretical displacement occurring at $\mathbf{X}$ if microcracks are absent:

$$\mathsf{u}(\mathbf{X}) = \mathbf{u}_{\text{real}}(\mathbf{X}) - \mathbf{u}(\mathbf{X}). \tag{5.2}$$

As common in multifield theories, u is not affected by rigid translations of the observer, and this is because it is a relative microdisplacement, the mean relative displacement between the margins of the microcrack at $\mathbf{X}$.

For the kinematical microstructure introduced $\mathscr{M} \equiv \mathscr{V}$, the order parameter u is a vector as the microforces k and b, while the microstress T is a second-order tensor. Moreover for microcracks we can assume $\|\mathsf{u}\| \ll 1$.

By the physical meaning of the field u it is possible to introduce the total deformation gradient

$$\mathbf{F}_t = \nabla_{\mathbf{X}}\mathbf{x} = \mathbf{I} + \nabla\mathbf{u} + \nabla\mathsf{u} = \widetilde{\mathbf{F}}\mathbf{F}, \tag{5.3}$$

where

$$\widetilde{\mathbf{F}} = \mathbf{I} + \nabla\mathsf{u}\mathbf{F}^{-1} \tag{5.4}$$

is the gradient of the additional deformation produced by microcracks, which we think superposed to that associated to $\mathbf{u}$. In this section the gradient of the director u is to be interpreted in the generalized sense.

It is also possible to introduce global measures of deformation. In the case of the linear theory with cracks which do not grow and evolve, assuming that $\|\nabla\mathbf{u}\| \ll 1$ and $\|\nabla\mathsf{u}\| \ll 1$, the overall (total) linearized strain tensor has an expression like the one used in infinitesimal plasticity:

$$\mathbf{E}_{\mathrm{t}} = \mathbf{E} + \mathsf{E}, \tag{5.5}$$

where

$$\mathsf{E} = \operatorname{sym} \nabla \mathsf{u}. \tag{5.6}$$

We assume null volume microforces, $\mathsf{b} = 0$, conditions (2.2) to be valid and, without loss of generality, homogeneous boundary conditions for the substructural fields, in particular $\mathsf{u}_0 = 0$ on $\partial_1 B$, which is equivalent to suppose no microcracks on the boundary. As discussed in Mariano (1999), this condition is generic: it seems natural to have a (macro-) displacement boundary condition different from zero, while any microcrack which arrives on the boundary will influence the roughness of the interested surface.

We also assume a linear elastic behavior of the microcracked body. Moreover, as it is customary in classical and modern theories of fracture mechanics (see Del Piero and Truskinovsky, 2001), where the elastic energy is split into a bulk part, which is a function of strain, and a surface (cohesive) part, which is a function of the components of relative displacements on the surface of discontinuity, we assume for the internal energy density the following decomposition:

$$\hat{w}(\mathbf{E}, \mathsf{u}, \nabla \mathsf{u}) = \varpi(\mathbf{E}, \nabla \mathsf{u}) + \theta(\mathsf{u}), \tag{5.7}$$

i.e. we assume the existence of an interfacial energy which is a function of the jump of the real displacement, in the present context a function of the order parameter u, the mean of the jumping part of the real displacement field. Such an assumption implies in the linearized constitutive relations (2.7a)–(2.7d) $\mathbf{h} = \mathbf{f} = 0$:

$$\begin{aligned} \mathbf{T} &= \mathbf{C}[\mathbf{E}] + \mathbf{H}[\nabla \mathsf{u}], \\ \mathsf{k} &= -\mathbf{K}\mathsf{u}, \\ \mathsf{T} &= \mathbf{S}[\nabla \mathsf{u}] + \mathbf{H}^{\mathrm{T}}[\mathbf{E}], \end{aligned} \tag{5.8}$$

the corresponding energy densities, respectively the bulk and the interfacial, having the forms

$$\varpi(\mathbf{E}, \nabla \mathsf{u}) = \tfrac{1}{2}(\mathbf{C}[\mathbf{E}] \cdot \mathbf{E} + \mathbf{S}[\nabla \mathsf{u}] \cdot \nabla \mathsf{u}) + \mathbf{H}[\nabla \mathsf{u}] \cdot \mathbf{E}, \tag{5.9}$$

$$\theta(\mathsf{u}) = \frac{1}{2}\mathbf{K}\mathsf{u} \cdot \mathsf{u}; \tag{5.10}$$

constitutive equations (5.8) can be arrived at by following the procedure of Mariano and Stazi (2001).

The obtained equilibrium problem in B with constitutive equations (5.8) is

$$\operatorname{div}(\mathbf{C}[\mathbf{E}] + \mathbf{H}[\nabla \mathsf{u}]) + \mathbf{b} = 0, \tag{5.11}$$

$$\operatorname{div}(\mathbf{S}[\nabla \mathsf{u}] + \mathbf{H}^{\mathrm{T}}[\mathbf{E}]) + \mathsf{b} - \mathbf{K}\mathsf{u} = 0; \tag{5.12}$$

the solution of (5.11) and (5.12) allows to obtain the localization phenomena of the deformation, thanks to the presence of the interactive microforce k which physically represents the interactions between the microcracks. This result is impossible to obtain in the standard linear elasticity model of Cauchy continua or in the elastic case of an internal variable model.

From the functional (2.14) we have the following mixed variational characterization of equilibrium problem for a body with microcracks:

$$\begin{aligned} \delta\mathfrak{F}\{\mathbf{u}, \mathbf{E}, \mathbf{T}, \mathsf{u}, \nabla \mathsf{u}, \mathsf{T}\} = \delta\bigg\{ \int_B [\varpi(\mathbf{E}, \nabla \mathsf{u}) + \theta(\mathsf{u}) + \mathbf{T} \cdot (\operatorname{sym} \nabla \mathbf{u} - \mathbf{E}) - \mathbf{b} \cdot \mathbf{u}] - \int_{\partial_1 B} (\mathbf{T}\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_0) + \mathsf{T}\mathbf{n} \cdot \mathsf{u}) \\ - \int_{\partial_2 B} (\mathbf{s}_0 \cdot \mathbf{u} + \mathsf{s}_0 \cdot \mathsf{u}) \bigg\} = 0 \end{aligned} \tag{5.13}$$

while from (4.6) we obtain the following linear version of the variational characterization first proposed by Mariano (1999) for nonlinear elasticity of microcracked bodies:

$$\delta\mathfrak{E}\{\mathbf{u}, \mathrm{u}, \nabla\mathrm{u}\} = \delta\left\{\int_B (\breve{\varpi}(\nabla\mathbf{u}, \nabla\mathrm{u}) + \theta(\mathrm{u}) - \mathbf{b}\cdot\mathbf{u}) - \int_{\partial_2 B} (\mathbf{s}_0\cdot\mathbf{u} + \mathrm{s}_0\cdot\mathrm{u})\right\} = 0,$$
$$\mathbf{u} - \mathbf{u}_0 = 0, \mathrm{u} = 0 \quad \text{on } \partial_1 B, \tag{5.14}$$

with

$$\breve{\varpi}(\nabla\mathbf{u}, \nabla\mathrm{u}) = \frac{1}{2}(\mathsf{C}[\nabla\mathbf{u}]\cdot\nabla\mathbf{u} + \mathsf{S}[\nabla\mathrm{u}]\cdot\nabla\mathrm{u}) + \mathsf{H}[\nabla\mathrm{u}]\cdot\nabla\mathbf{u}; \tag{5.15}$$

these variational characterizations are useful to derive numerical formulations and solutions of the differential system (5.11) and (5.12) (see Mariano and Stazi, 2001, where a finite element formulation in the spirit of the displacement method was worked out).

6. Application II. Piezoelectric bodies

Here we deal with the linear theory of piezoelectricity. In particular, we want to set the theory of piezoelectric continua in the multifield theories context, as a particular case of continua with a scalar microstructure characterized by a null interactive microforce k.

As it is well known, in the linear theory of piezoelectric bodies (valid in the case of infinitesimal mechanical deformations and small electric fields), the equilibrium equations are (2.3a), (2.3d) and (2.13), while the kinematical equations are (2.4a) and (2.4b). Moreover, let $\mathbf{e}(\mathbf{X}, t)$ and $\mathbf{d}(\mathbf{X}, t)$ denote respectively the *electric* and *electric displacement* fields over B, which obey the Maxwell equations in the quasi-static approximation (see Maugin, 1988)

$$\operatorname{curl}\mathbf{e} = 0, \quad \text{or} \quad \mathbf{e} = -\nabla\phi \quad \text{in } B, \tag{6.1}$$

$$\operatorname{div}\mathbf{d} = \omega \quad \text{in } B, \tag{6.2}$$

together with the associated boundary conditions

$$\mathbf{d}\cdot\mathbf{n} = q \quad \text{on } \partial_q B, \tag{6.3}$$

$$\phi = \phi_0 \quad \text{on } \partial_\phi B; \tag{6.4}$$

here ϕ is the electric potential, ω is the volume charge density defined on B, q is the surface charge density prescribed on the portion of the boundary $\partial_q B$ and ϕ_0 is the prescribed electric potential on the complementary part of the boundary $\partial_\phi B$. Moreover, the infinitesimal strain, the stress, the electric and electric displacement fields are related by the linear constitutive equations (see Voigt, 1910):

$$\mathbf{T} = \mathsf{C}^{(\mathrm{e})}[\mathbf{E}] - \mathsf{h}[\mathbf{e}],$$
$$\mathbf{d} = \mathsf{h}^{\mathrm{T}}[\mathbf{E}] + \mathbf{K}^{(\mathrm{E})}\mathbf{e}, \tag{6.5}$$

where the *elasticity tensor* field $\mathsf{C}^{(\mathrm{e})}$, evaluated at constant electric field $\mathbf{e}$, is a symmetric, positive-definite linear mapping from Sym to itself, the *dielectric tensor* field $\mathbf{K}^{(\mathrm{E})}$, evaluated at constant mechanical strain $\mathbf{E}$, is a symmetric positive-definite linear mapping of Lin to Lin and the *piezoelectric stress tensor* field h is a linear mapping from $\mathscr{V}$ to Sym, whose transpose h^{T} maps Sym into $\mathscr{V}$ (in components, $(\mathsf{h}^{\mathrm{T}})_{ijk} = \mathsf{h}_{kij}$). So a *piezoelastic state* is described by the pair $(\mathbf{E},\mathbf{e})$ or equivalently, because of the (6.1, part 1), by the pair $(\mathbf{E}, \phi)$.

If we assume the electric potential $\phi(\mathbf{X})$ as the order parameter, we are in the case of a scalar microstructure, with $\mathscr{M} \equiv \mathbb{R}$; by means of the Maxwell equation (6.1, part 2), in this context a kinematic equation, the order parameter gradient $\nabla\phi$ represents the opposite of the electric field $\mathbf{e}$ and the dual quantity, the microstress T is a vector playing the role of the electric displacement field $\mathbf{d}$.

Since in a piezoelectric body the total energy density can be written as a function of the deformation and of the electric field, $\chi = \hat{\chi}(\mathbf{E}, \mathbf{e})$, we assume that the total energy density (2.6) is not explicitly dependent on the order parameter itself ϕ

$$w = \hat{w}(\mathbf{E}, \nabla\phi) \tag{6.6}$$

and that has the quadratic representation:

$$\hat{w}(\mathbf{E}, \nabla\phi) = \tfrac{1}{2}(\mathsf{C}[\mathbf{E}] \cdot \mathbf{E} + \mathbf{S}[\nabla\phi] \cdot \nabla\phi) + \mathsf{h}[\nabla\phi] \cdot \mathbf{E}. \tag{6.7}$$

As consequence, constitutive equations for the general linear theory of an hyperelastic continuum with a scalar microstructure reduce to (6.5), with h the third-order piezoelectric stress tensor and $\mathbf{S}$ the opposite of the second-order symmetric dielectric tensor. Moreover, the interactive microforce is identically zero $\mathsf{k} = -\partial w/\partial\phi = 0$ and the Maxwell equation (6.2) can be obtained as a balance of substructural interactions by means of (2.3b), with the volume microforce b (which in the case of a scalar microstructure is a scalar field on B) equals to the opposite of the volume charge density ω.

By applying the mixed functional (4.8), identifying $\partial_\phi B$ with $\partial_3 B$ and $\partial_q B$ with $\partial_4 B$, we obtain the following six fields functional for the variational characterization of the equilibrium problem of a linear piezoelectric body:

$$\begin{aligned}\mathfrak{Y}\{\mathbf{u}, \mathbf{E}, \mathbf{T}, \phi, \nabla\phi, \mathbf{d}\} = \int_B [w(\mathbf{E}, \nabla\phi) + \mathbf{T} \cdot (\operatorname{sym}\nabla\mathbf{u} - \mathbf{E}) - \mathbf{b} \cdot \mathbf{u} + \omega\phi] - \int_{\partial_1 B} \mathbf{Tn} \cdot (\mathbf{u} - \mathbf{u}_0) \\ - \int_{\partial_2 B} \mathbf{s}_0 \cdot \mathbf{u} - \int_{\partial_\phi B} \mathbf{d} \cdot \mathbf{n}(\phi - \phi_0) - \int_{\partial_q B} q\phi. \end{aligned} \tag{6.8}$$

The functional (6.8) coincides with the functional obtained in He (2001, cf. Eq. (4.39)), with the constraint $\mathbf{e} = -\nabla\phi$ in B, and it is the simplified version of the Hu–Washizu type functional obtained by Yang and Batra for the nonlinear theory of electroelasticity (see Yang and Batra, 1995, Eq. (29)), in the case of quasi-static deformations of a linear piezoelectric body.

7. Conclusions

We have derived a Hu–Washizu-like variational principle for continua with vectorial microstructure, that is continua endowed with a director field, using a semi-inverse approach instead of the classical Lagrange multipliers method. Then we have applied the obtained six fields mixed functional, valid in a fully linear setting, to microcracked and piezoelectric bodies, obtaining in both cases variational principles useful for applications.

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